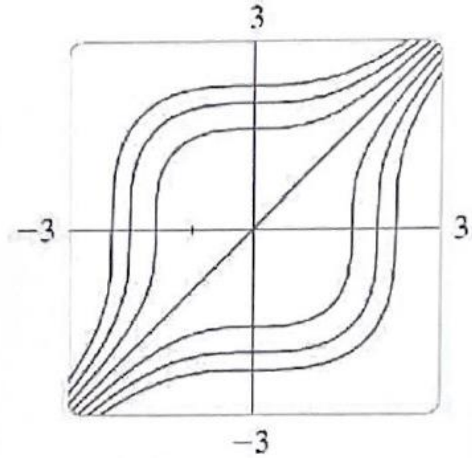
1. The graph of several solutions to the differential equation $\frac{dy}{dx}=\frac{x^2}{y^2}$ is shown. Solve the equation, then find the particular solution that satisfy the initial conditions (a) $y(0)=2$, (b) $y(0)=-2$, and (c) $y(0)=0$.



$\frac{dy}{dx} = \frac{x^2}{y^2}$

$y^2\,dy = x^2\,dx$

$\frac{1}{3}y^3 = \frac{1}{3}x^3 + C$

$y^3 = x^3 + C$

$y = \sqrt[3]{x^3+C}$

a) $2 = \sqrt[3]{0+C}$

$C = 8$

$y = \sqrt[3]{x^3+8}$

b) $-2 = \sqrt[3]{0+C}$

$-8 = C$

$y = \sqrt[3]{x^3-8}$

c) $0 = \sqrt[3]{0+C}$

$0 = C$

$y = \sqrt[3]{x^3}$

$y = x$

2. Find the general and particular solutions to the separable differential equation $\frac{dy}{dx}=x^2y$ given the initial conditions (a) $f(0)=1$ and (b) $f(0)=-2$.

$\frac{dy}{dx} = x^2y$

$\frac{1}{y}\,dy = x^2\,dx$

$\ln|y| = \frac{1}{3}x^3 + C$

$|y| = e^{\frac{1}{3}x^3+C}$

$y = Ce^{\frac{1}{3}x^3}$

a) $1 = Ce^0$

$1 = C$

$y = e^{\frac{1}{3}x^3}$

b) $-2 = Ce^0$

$-2 = C$

$y = -2e^{\frac{1}{3}x^3}$

3. Let f be a function with $f(1) = 4$ such that for all points (x, y) on the graph of f, the slope is given by

$$\frac{dy}{dx} = \frac{3x^2+1}{2y}.$$

(a) Find the slope of the graph of f at the point where $x = 1$.

$$\left.\frac{dy}{dx}\right|_{(1,4)} = \frac{4}{8} = \frac{1}{2}$$

(b) Write an equation of the line tangent to the graph of f at $x = 1$ and use it to approximate $f(1.2)$.

$$y - 4 = \frac{1}{2}(x-1)$$

$$L(x) = 4 + \frac{1}{2}(x-1)$$

$$L(1.2) = 4 + \frac{1}{2}(0.2) = 4.1$$

$$\underline{f(1.2) \approx 4.1}$$

(c) Find $f(x)$ by solving the separable differential equation $\frac{dy}{dx} = \frac{3x^2+1}{2y}$ with the initial condition $f(1) = 4$.

$$2y\,dy = (3x^2+1)\,dx$$

$$y^2 = x^3 + x + C$$

$$y = \pm\sqrt{x^3+x+C}$$

$$4 = \sqrt{1+1+C}$$

$$16 = 2 + C$$

$$14 = C$$

$$\underline{y = \sqrt{x^3+x+14}}$$

(d) Use your solution from part (c) to find the exact value of $f(1.2)$.

$$f(1.2) = \sqrt{(1.2)^3 + 1.2 + 14} = \underline{4.114}$$

4. Consider the differential equation $\frac{dy}{dx} = 6 - 2y$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 4$.

(a) Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 0$. Use the tangent line to approximate $f(0.6)$.

$\left.\frac{dy}{dx}\right|_{(0,4)} = -2$

$y - 4 = -2(x - 0)$

$L(x) = 4 - 2x$

$L(0.6) = 4 - 2(0.6)$

$= 4 - 1.2$

$= 2.8$

$\boxed{f(0.6) \approx 2.8}$

(b) Find $\frac{d^2y}{dx^2}$. Is the approximation found in part (a) an overestimate or underestimate of the actual value of $f(0.6)$. Justify your answer.

$\frac{d^2y}{dx^2} = -2\frac{dy}{dx}$

$\left.\frac{d^2y}{dx^2}\right|_{(0,4)} = -2(-2) = 4 > 0$

y is concave up

Since $\frac{d^2y}{dx^2} > 0$

$L(0.6) < f(0.6)$

because the tangent line is below a curve that is concave up.

(c) Find $y = f(x)$, the particular solution to the differential equation with the initial condition $f(0) = 4$.

$\frac{dy}{dx} = 6 - 2y$

$\frac{1}{6-2y}\,dy = dx$

$\frac{1}{-2(y-3)}\,dy = dx$

$\frac{1}{y-3}\,dy = -2\,dx$

$\ln|y-3| = -2x + C$

$|y-3| = e^{-2x+C}$

$y - 3 = Ce^{-2x}$

$y = 3 + Ce^{-2x}$

$f(0) = 4$

$4 = 3 + Ce^{0}$

$1 = C$

$\boxed{f(x) = 3 + e^{-2x}}$

(d) For the particular solution $y = f(x)$ found in part (c), find $\lim_{x\to\infty} f(x)$.

$\lim_{x\to\infty}\left[3 + e^{-2x}\right] = 3 + 0 = \underline{3}$